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Exact statistical properties of a model for interacting waves in non-thermal equilibrium

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Abstract. A particular model for N coupled waves in non-thermal equilibrium is studied in detail. Exact closed forms for the one- and two-wave intensity distribution functions and an exact recurrence relation for the photon counting distribution function are obtained. The 'thermodynamic limit', $N \to \infty$, is discussed with special reference to the multi-mode laser, and a mean field type phase transition is shown to occur at laser threshold.

1. Introduction

It has been known for some time, how, starting from the exact quantum-mechanical Liouville equation for a closed system, consisting of all physically relevant quantities, one can derive an exact c number equation for a quasi-distribution function which is still a function of all the relevant variables.

This quasi-classical correspondence, which in recent years has been considered in detail in conjunction with the laser (cf Lax 1968, Haken 1970), allows one to express the statistical problems associated with non-thermal flow equilibria into a framework similar to that of equilibrium statistical mechanics. The relevant equation may be considerably simplified by restricting ones attention to an open sub-system and treating the interaction with the rest of the system in some approximate manner. This procedure usually leads, in a reasonable approximation, to a Fokker–Planck (FP) equation for a continuous Markov process which may be written in the standard form (Lax 1966)

$$\frac{\partial}{\partial t}P = -\sum_i \frac{\partial}{\partial x_i}[A_i(\{x\})P] + \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}[D_{ij}(\{x\})P],$$

where A is the drift vector, D the diffusion matrix, and x_i are real phase space variables.

The stationary solution of this FP equation then plays the same role for the static statistical properties of the system as the Gibbs canonical distribution for thermal equilibrium. It may be written in the form $P \propto \exp[-V]$, where V is a potential function involving all phase space variables and depends on the form of the drift and diffusion coefficients of the underlying FP equation. Under the condition of detailed balance, one can in fact construct V uniquely from the drift and diffusion terms if the generalized potential conditions devised by Graham and Haken (1971) are fulfilled. If, in addition, the system is non-oscillatory V is obtained in simple cases by direct line integration of the drift vectors (Lax 1966). This case is of special interest in that there exists an intimate relation between V and the dynamics of the system contained in the drift vector (Graham 1973). Finally attention is drawn to a novel method proposed by Haken (1973) to construct V from the constants of motion of a truncated FP equation. One may therefore assume that V and consequently the stationary distribution function is known, leaving as a major problem the evaluation of physically interesting quantities that may be derived from this form for P.

For an interacting multi-wave system, like a multi-mode laser, the variables of the problem are the amplitudes and phases of the waves. Thus P is a distribution function of many variables and of little direct practical use unless one can use it to compute reduced distribution functions say for one or two waves or just for the intensity of one wave. This reduction process involves integrations over the variables which are not of interest in complete analogy to equilibrium statistical mechanics. These integrations can be done only for simple systems with specially chosen interactions.

Since actually little is known about the statistical properties of interacting waves in non-thermal equilibrium it seems worthwhile to study solvable models in detail.

A tractable model of many interacting waves has been introduced by Wonneberger and Lempert (1973a, to be referred to as WL). In WL it was shown, using an asymptotic treatment appropriate to a large number, N, of waves, that the beta distribution of mathematical statistics (cf eg Maisel 1971) accounts for the statistical properties of the model.

Since then it has been found that a much more comprehensive treatment of this model is possible which allows one to exploit all static properties in a virtually exact way. Especially, it has become possible to investigate the 'thermodynamic limit', $N \to \infty$, that is, a system of infinitely many interacting waves. Such a system is expected to show critical behaviour similar to that recently discovered theoretically in the system of N atoms interacting with a single light mode (Dohm 1972a, b, Haken and Wöhrstein 1973).

In § 2, we introduce the potential and discuss its physical relevance. Section 3 is devoted to the mathematics of the problem, and the general form of the reduced distribution functions, the generating function and the photon counting distribution are derived. In § 4 the asymptotic regime investigated in WL is discussed and the range of validity of these earlier results assessed.

Sections 5 and 6 deal with detailed predictions derived from the one- and two-wave distribution functions. Finally, in § 7 the thermodynamic limit is considered and its relation to a phase transition is studied. The implications for a possible phase transition in a coupled multi-wave system, for example that of a multi-mode laser, are discussed in detail.

2. Choice of potential

The potential to be studied in the following is given by

$$V = \frac{1}{4}\left(\sum_{\nu=1}^{N} (z(\nu)-a)\right)^2, \tag{1}$$

where $z(\nu)$ are the scaled intensities of the waves numbered by indices ν and a is the pumping parameter specifying the state of the system.

For $N = 1$, this potential was first derived by Risken (1965) from the stochastic rotating wave Van der Pol oscillator in his theory of the statistical properties of the single-mode laser. The reader is referred to Risken (1970) and to appendix 1 in the present paper for the notions of scaling and pumping parameter. The multi-wave potential of the form of equation (1) is appropriate to randomly phased waves fluctuating about an operating point and confined to a small frequency band (cf WL). This form of V is expected to be fairly realistic for the non-resonant feedback laser (Ambartsumyan *et al* 1967, cf also Brunner and Paul 1969) as well as for highly amplified vibrational flux in piezoelectric semiconductors (Wonneberger 1971).

The principle feature of V is that every wave interacts with itself and with all other waves with the same coupling strength. This is called the neutral coupling case in a multi-mode laser (Sargent *et al* 1967). It leads to the characteristic dependence of V on $\Sigma\, z(\nu)$ only, which is the formal reason for the tractability of the model.

Another way of arriving at this form of V is sketched in appendix 1, namely from the viewpoint of the presence of an instability mechanism. The wave system becomes unstable when the pumping parameter, a, changes sign from minus to plus. A non-linear feedback mechanism involving all waves eventually drives the system into a new stable state far from thermal equilibrium.

However even when the form for V has been decided the major problem still remains. It may be stated in the following manner. Given values for the pumping parameter a and the total number N of interacting waves, what are the distribution functions of $z(1)$, $(z(1), z(2))$ etc and of the incoherent superimposition I of N'' $(<N)$ wave intensities? The latter quantity is of practical interest in scattering experiments (Wonneberger *et al* 1974). In the next section an exact answer will be given to these questions for the particular form of the potential discussed above.

3. Mathematical procedure

In the notation of WL we define a reduced distribution function

$$P_{\mathrm{c}}^{(N'')}(z(1), \ldots, z(N'')) \triangleq \int_0^\infty \prod_{\nu=N''+1}^{N} \mathrm{d}z(\nu) \exp(-V),$$

which for the particular potential given by equation (1) may be expressed in the form

$$P_{\mathrm{c}}^{(N'')}(z(1), \ldots, z(N''))$$
$$\triangleq \exp[-\tfrac{1}{4}(I-Na)^2] \int_0^\infty \mathrm{d}x\, x^{N'-1} \exp[-\tfrac{1}{2}(I-Na)x - \tfrac{1}{4}x^2], \tag{2}$$

where $I = \Sigma_{\mu=1}^{N''} z(\mu)$, and $N' = N - N'' \geqslant 1$. The $\triangleq$ sign indicates that P_{c} is not normalized. This integral was evaluated in WL using the method of steepest descent. However the accuracy of this approximation is difficult to assess. Fortunately this difficulty may be circumvented since it is possible to evaluate the integral exactly in closed form. In fact $P_{\mathrm{c}}^{(N'')}$ may be expressed in terms of parabolic cylinder functions, which also appear in the corresponding single-wave treatment (Risken 1970), giving

$$P_{\mathrm{c}}^{(N'')} \triangleq \exp[-\tfrac{1}{8}(I-Na)^2] D_{-N'}\left(\frac{I-Na}{\sqrt{2}}\right). \tag{3}$$

Note that P_{c} is a function of I only. This formal solution is of little practical use except for the calculation of the low-order distribution functions $P_1(z(1))$ and $P_2(z(1), z(2))$ (§§ 5 and 6).

However, as shown below, $P_{\mathrm{c}}^{(N'')}$ satisfies an ordinary second-order differential equation in the variable I, for any N''. This then allows one to study the behaviour of the distribution function $P_c^{(N'')}$ in a simple way. The differential equation can either be obtained by straightforward differentiation of equation (2) with respect to I and subsequent determination of the generated distributions $P_c^{(N''-1)}$ and $P_c^{(N''-2)}$ by partial integrations or more formally by differentiating equation (3) and using the recurrence relations satisfied by $D_{-p}(z)$. The result is simply

$$2\frac{\mathrm{d}^2}{\mathrm{d}I^2}P_c^{(N'')}(I) = (N'-1)P_c^{(N'')}(I)+(Na-I)\frac{\mathrm{d}}{\mathrm{d}I}P_c^{(N'')}(I). \tag{4}$$

This equation also applies for $N' = 0$ in which case $I = z$, and it then describes the single-mode case with an effective pump parameter equal to Na.

The distribution function

$$P(I) = \int_0^\infty \prod_{\mu=1}^{N''} \mathrm{d}z(\mu)\delta\left(I - \sum_{\mu'=1}^{N''} z(\mu')\right) P_c^{(N'')}(\{z\}),$$

is easily found by employing the contour integration method used in WL and one obtains

$$P(I) \triangleq I^{N''-1}P_c^{(N'')}(I).$$

The differential equation satisfied by $P(I)$ follows from that for P_c and reads:

$$\begin{aligned}2I^2\frac{\mathrm{d}^2}{\mathrm{d}I^2}P(I) &= [4(N''-1)+(Na-I)I]I\frac{\mathrm{d}}{\mathrm{d}I}P(I)\\ &\quad +[(N'-1)I^2-(Na-I)(N''-1)I-2(N''-1)N'']P(I).\end{aligned} \tag{5}$$

A quantity of central interest is the generating function

$$Q(\lambda) = \int_0^\infty \mathrm{d}I \exp(-\lambda I)P(I).$$

By Laplace transformation of equation (5) one obtains the third-order differential equation

$$\lambda Q'''(\lambda)+(N+1+Na\lambda-2\lambda^2)Q''(\lambda)+(N''+1)(Na-4\lambda)Q'(\lambda)-2N''(N''+1)Q(\lambda) = 0. \tag{6}$$

The boundary-value problem associated with this equation is solved by noting that

$$\begin{aligned}Q(\lambda = 0) &= 1,\\ Q'(\lambda = 0) &= -\langle I\rangle = -N''\langle z\rangle,\\ Q''(\lambda = 0) &= \langle I^2\rangle = N''\langle z^2\rangle+(N''-1)N''\langle z(1)z(2)\rangle.\end{aligned} \tag{7}$$

Here, use has been made of the equivalence of the waves. $\langle z\rangle$, $\langle z^2\rangle$ and $\langle z(1)z(2)\rangle$, are the various moments of P and are explicitly determined by equations (12), (13) and (17) below.

Following the procedure in WL, one can obtain from equation (6) a recurrence relation for the photon counting distribution $p(n)$ defined by (Mandel 1958)

$$p(n) = \frac{1}{n!}\left(-\nu_1\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^n Q(\lambda)\bigg|_{\lambda=\nu_1}.$$

For a competent review of photon statistics the reader is referred to Pike (1970). The above expression is appropriate for short counting intervals $T \ll \tau_c$, where τ_c is the intensity correlation time and ν_1 is related to the quantum efficiency of the absorption photon detector. The result of a simple but lengthy calculation is the four stage recurrence relation

$$\begin{aligned}(n+1)(n+2)(n+3)p(n+3)-(n+N+1+Na\nu_1)(n+1)(n+2)p(n+2)\\ +(n+N''+1)Na\nu_1(n+1)p(n+1)\\ = -2\nu_1^2\{(n+1)(n+2)p(n+2)-2(n+N''+1)(n+1)p(n+1)\\ +[n-1+(N''+1)(N''+2)]np(n)\}. \qquad (8)\end{aligned}$$

In the limit $\nu_1^2 \to 0$ the left-hand side of equation (8) is equivalent to the form obtained in WL. Equation (8) lends itself easily to computer evaluation.

This completes the main formal results, which are an exact consequence of the potential being of the form given by equation (1).

4. An asymptotic result

In WL an asymptotic regime has been investigated which is roughly characterized by N large, $a > 0$ but not too small. The precise conditions for the validity of this asymptotic regime will now be established. We transform equation (6) to the new variable $x = -Na\lambda$ and obtain:

$$\begin{aligned}2x^2\ddot{Q}(x)+4(N''+1)x\dot{Q}(x)+2N''(N''+1)Q(x)\\ = (Na)^2[x\dddot{Q}(x)+(N+1-x)\ddot{Q}(x)-(N''+1)\dot{Q}(x)], \qquad (9)\end{aligned}$$

where a dot denotes differentiation with respect to x. For large enough values of $(Na)^2$, the term in the square brackets must vanish. This gives immediately a solution in terms of a confluent hypergeometric function:

$$Q(\lambda) = {}_1F_1(N'', N, -\lambda Na), \qquad (10)$$

which is the asymptotic result obtained in WL. The associated distribution $P(I)$ is a beta distribution again as found in WL. A comparison of the magnitudes of the different terms in equation (9) under the assumption that $Q(\lambda)$ is of the form given by equation (10) gives the following result. For $a > 0$ and $Na^2 \gg 1$ and for λ values in the range $0 \leqslant \lambda \lesssim a^{-1}$ one has only to retain those terms in the full equation (9) which actually lead to equation (10). The photon counting distribution (8) requires $\lambda = \nu_1$. If equation (10) is valid ν_1 is related to the mean count number $\langle n\rangle$ by $\langle n\rangle = N''a\nu_1$ (cf WL). This gives a bound $\langle n\rangle \lesssim N''$ to the applicability of the counting distribution, equation (II2) in WL, in the sense of arising from the form of the potential V, and under the conditions $Na^2 \gg 1$, $a > 0$. If $a^2 \gg 1$ the weaker and usually sufficient condition $\langle n\rangle \lesssim N''N$ is obtained.

In summary it has been shown that the distribution $P(I)$ takes the form of a beta distribution in the asymptotic limit defined by $a > 0$ and $Na^2 \gg 1$, irrespective of the values of N'' and N'. If N is small but $a^2 \gg 1$, equation (10) is still a valid representation of $Q(\lambda)$. Thus the application of the beta distribution to experiments in Wonneberger and Lempert (1973b) and Wonneberger *et al* (1974) is fully justified.

5. One-wave distribution

We now begin to investigate the results of § 3 which are not covered by the asymptotic limit given by the beta distribution.

An important class of problems is associated with $P_1(z)$, that is with the distribution function for the intensity of one wave coupled to the remaining $N-1$ waves. By direct computation involving the properties of the parabolic cylinder functions (cf Abramowitz and Stegun 1964) one obtains explicitly

$$P_1(z) = \frac{1}{\sqrt{2}} \exp(\tfrac{1}{4}Naz - \tfrac{1}{8}z^2) \frac{D_{-(N-1)}((z-Na)/\sqrt{2})}{D_{-N}(-Na/\sqrt{2})}. \tag{11}$$

For $N = 1$ this reduces to the result obtained by Risken (1965).

Direct evaluation, using equation (11), gives, for the mean intensity

$$\langle z \rangle = \int_0^\infty \mathrm{d}z\, zP_1(z) = a + \frac{2}{N}P_1(0) \equiv \sqrt{2}\frac{D_{-(N+1)}(-Na/\sqrt{2})}{D_{-N}(-Na/\sqrt{2})}, \tag{12}$$

and for the mean squared intensity

$$\langle z^2 \rangle = \int_0^\infty \mathrm{d}z\, z^2 P_1(z) = 2\frac{2+Na\langle z \rangle}{N+1} \equiv 4\frac{D_{-(N+2)}(-Na/\sqrt{2})}{D_{-N}(-Na/\sqrt{2})}. \tag{13}$$

Higher moments may be computed by using equation (6) as the basis of a recurrence relation.

$P_1(z)$ has the remarkable property that it is a strictly decreasing function of z for all a and all $N > 1$. That is, regardless of the values of $N(>1)$ and a, there occurs no amplitude stabilization. The formal proof for this is given in appendix 2. This absence of amplitude stabilization is most probably a consequence of wave coupling. If one introduces the coupling strength ξ following Grossmann and Richter (1971b) as the value of all off-diagonal elements of the scaled coupling matrix (equation (A.2) with $c\ (\nu, \nu) = 1$ in appendix 1) then one finds the following behaviour. For every $N \geqslant 2$ and every a there exists a critical value $\xi_c(N, a) < 1$ such that for $\xi > \xi_c$ amplitude stabilization is absent. $\xi = \xi_c$ just allows a horizontal tangent in $P_1(z)$ at $z = 0$. In the neutral coupling case, $\xi = 1$ and since $\xi_c < 1$, amplitude stabilization is therefore strictly absent.

The differential equation for $P_1(z)$ (with $N' = N-1$) allows a very simple discussion of the behaviour of $P_1(z)$ and $\langle z \rangle$ in several limiting cases. We have seen above that $P_1(z)$ is a monotonically decreasing function so that for large z its value becomes insignificant. Let us now consider the case where $P_1(z)$ is negligible for all $z \gtrsim N|a|$. In that case one may neglect z compared to Na in equation (4) to obtain

$$2P_1'' = (N-2)P_1 + NaP_1'.$$

This has a solution $P_1(z) \propto \exp(kz)$, where k is given by

$$k = \frac{Na}{4} - \left[\left(\frac{Na}{4}\right)^2 + \frac{N-2}{2}\right]^{1/2} < 0.$$

This solution is consistent with the approximation made if $|kNa| \gg 1$. The important conclusion one can draw is that under this condition every wave is thermal and its mean value is given by

$$\langle z \rangle = -k^{-1}.$$

We may consider three separate regions of validity:

(i) $a > 0, N \gg 1$. If in addition $Na^2 \gg 1$ one has

$$\langle z \rangle = a + \frac{2}{Na}. \tag{14}$$

The second term is a small correction to the corresponding result associated with the beta distribution (§ 4).

(ii) $a < 0, Na^2 \gg 1$. Then one obtains irrespective of N

$$\langle z \rangle = \frac{2}{N|a|}. \tag{15}$$

This is the well known result for a linear gaussian process with damping constant $N|a|$ and scaled fluctuation strength $Q = 4$ (cf Risken 1970).

(iii) $Na^2 \ll 1, N \gg 1$ (threshold region). Here

$$\langle z \rangle = \left(\frac{2}{N}\right)^{1/2} + \frac{a}{2},$$

where the first term dominates the second because $Na^2 \ll 1$.

It is easily checked that in all three cases $N|a|$ is much greater than the relevant values of z which are of order $\langle z \rangle$, that is $|kNa| \gg 1$.

In summary, the three cases, which essentially refer to many-wave situations, lead to a thermal distribution $P_1(z)$ for each individual wave of the form

$$P_1(z) = \frac{1}{\langle z \rangle} \exp\left(-\frac{z}{\langle z \rangle}\right).$$

The behaviour of $\langle z \rangle$ as a function of a is sketched in figure 1. In particular we note that in the limit $N \to \infty$

$$\langle z \rangle = a\theta(a),$$

where

$$\theta(a) = \begin{cases} 1, & a > 0, \\ 0, & a < 0. \end{cases}$$

The slope of the curve at $a = 0$ is equal to $\frac{1}{2}$ for all finite values of N.

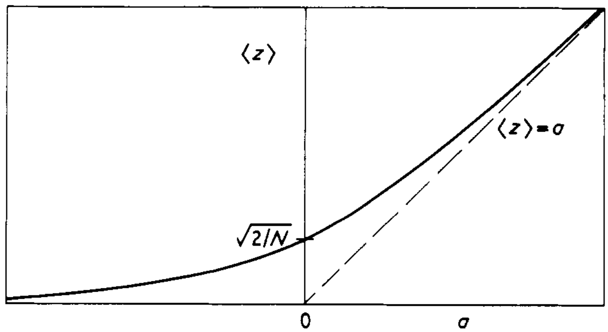

Figure 1. Mean intensity $\langle z \rangle$ of a wave coupled to $N-1$ ($\gg 1$) other waves as a function of the pumping parameter a.

6. Two-wave distribution

In this section we give a short account of some results associated with the two-wave distribution. Evaluation of the normalization integral associated with equation (3) for $N' = N-2$ gives

$$P_2(z(1), z(2)) = \tfrac{1}{2} \exp\left(\frac{Na}{4}(z(1)+z(2)) - \frac{(z(1)+z(2))^2}{8}\right) \frac{D_{-(N-2)}((z(1)+z(2)-Na)/\sqrt{2})}{D_{-N}(-Na/\sqrt{2})}. \quad (16)$$

From this, it may be shown that the cross correlations $\langle z(\mu)z(\nu \neq \mu)\rangle \equiv \langle z(1)z(2)\rangle$ are given by

$$\langle z(1)z(2)\rangle = \tfrac{1}{2}\langle z^2\rangle, \quad (17)$$

where $\langle z^2\rangle$ is given by equation (13).

Using the asymptotic formula for parabolic cylinder functions (cf Abramowitz and Stegun 1964)

$$D_{-p}(-|x|) \propto |x|^{(p-1)} \exp\left(\frac{|x|^2}{4}\right),$$

which is valid for $x < 0$, $x^2 \gg 4p \gtrsim 1$ one finds, under the corresponding conditions $a > 0$, $Na^2 \gg 1$, $N \gg 1$ that

$$P_2(z(1), z(2)) \propto \left(1 - \frac{z(1)+z(2)}{Na}\right)^{N-2},$$

which, for N large enough, reduces to

$$P_2 \propto \exp\left(-\frac{z(1)}{a}\right) \exp\left(-\frac{z(2)}{a}\right).$$

This latter result may also be obtained from equation (4) in the appropriate limit. This shows that any two waves are thermal and independent, hence uncorrelated. Absence of correlation seems to be a general feature for any small group of waves interacting with many waves. This lack of correlation implies, using equation (17), that

$$\langle z^2\rangle = 2\langle z(1)z(2)\rangle = 2\langle z\rangle^2, \quad (18)$$

which again shows the gaussian nature of the underlying statistics.

7. The thermodynamic limit $N \to \infty$

Early experimental observations especially on solid-state lasers (Phelan and Rediker 1965, Hunsperger and Ballantyne 1967) revealed a sharp onset of lasing operation when the pump strength reached threshold. Later there has been considerable theoretical speculation about the relation of this laser instability to a phase transition far from thermal equilibrium. Neglecting fluctuations, Degiorgio and Scully (1970) have established a one-to-one correspondence between a single-mode optical maser and a ferromagnet described by the Curie–Weiss theory. The light-field amplitude is the order parameter and the inversion plays the role of the temperature. Graham and Haken

(1970) have cast the problem of the condensation of all thermal cavity modes into a lasing non-thermal mode during the passage through threshold, into the framework of the phenomenological Landau theory of phase transitions (cf also Grossmann and Richter 1971a). In these treatments, the thermodynamic limit, which is essential to the mathematical theory of phase transitions, plays no role. Recently Dohm (1972a, b) has solved exactly the laser equations appropriate to a single light mode coupled to N two-level atoms. He has shown that in the limit $N \to \infty$ the atomic polarization per atom exhibits a mean-field phase transition with the pumping parameter behaving as the effective temperature. Identical conclusions have also been obtained by Haken and Wöhrstein (1973).

The light-field mode shows the same behaviour in Dohm's theory if the quantity $(\langle n\rangle/N)^{1/2}$ is taken as the order parameter p. Here, $\langle n\rangle$ is the photon population of the mode. However it will be noted that this quantity diverges in the limit $N \to \infty$.

Identifying p with $\langle z\rangle^{1/2}$, we find exactly the same mean-field phase transition behaviour in our model of N interacting waves in the limit $N \to \infty$. In this limit, as is shown in § 5, we may write

$$p = a^{1/2}\theta(a), \tag{19}$$

giving the critical index $\beta = 1/2$.

Indeed, a corresponds to the reduced temperature ϵ used in the description of equilibrium phase transitions. This is the expected mean-field behaviour and arises because of the long-range nature of the interaction in equation (1). The equivalent to the specific heat is the derivative of $\langle z\rangle$ with respect to a, which evidently has critical exponents $\alpha = \alpha' = 0$ (jump discontinuity). Consequently one expects $\gamma = 1$ for the critical exponent of the equivalent to the compressibility (cf Stanley 1971). This is the linear response of the wave field to an external field of the same frequency. From the work of Agarwal (1972) on linear responses for non-thermal equilibria it is known that these generalized susceptibilities are related by dissipation fluctuation theorems to the spectral functions of the underlying Markov process. For the static case (frequency of injected signal = frequency of wave field) the susceptibility simplifies considerably and may be expressed by static averages alone. In detail, in the notation of Agarwal (1972), we choose the response operator to be

$$B = \sum_{\nu=1}^{N} x_\nu,$$

where, without loss of generality, $x_\nu = \mathrm{Re}\,\tilde{u}(\nu)$ and $\tilde{u}(\nu)$ is the complex amplitude of the wave ν, that is $z(\nu) = \tilde{u}(\nu)\tilde{u}^*(\nu)$. This situation corresponds to the injection of a signal with fixed phase (coherent signal). The static response $\delta\langle B\rangle$ is then described by a susceptibility

$$\chi_N(a, \omega = 0) = \sum_{\nu,\nu'=1}^{N} \langle x_\nu x_{\nu'}\rangle,$$

which gives

$$\chi(a) = \lim_{N\to\infty} (\tfrac{1}{2}N\langle z\rangle), \tag{20}$$

because the distribution function in equilibrium is phase invariant and $\langle x_\nu^2\rangle = \frac{1}{2}\langle z\rangle$.

Using the result given by equation (15), which is appropriate below threshold, but because of the thermodynamic limit also true arbitrarily near to it, we obtain:

$$\chi(a < 0) = \frac{1}{|a|}. \tag{21}$$

This is the equivalent to the Curie–Weiss law for the susceptibility of a ferromagnet in its paramagnetic phase. The critical exponent associated with χ is $\gamma = 1$, thus confirming the mean-field nature of the phase transition in this exactly soluble model. For $a > 0$, that is above threshold, χ is infinite, indicating the simultaneous phase ordering of all waves if an external field is applied.

Though our results from the phase transition point of view are qualitatively the same as those of Degiorgio and Scully (1970), they are obtained by taking the thermodynamic limit and not by neglecting fluctuations. In fact, neglect of fluctuations is completely inadmissible in our case, where each wave is thermal and thus its intensity undergoes large fluctuations (cf equation (18)).

Some remarks concerning the relation of onset of lasing operation in a laser to that of spontaneous order in a second-order phase transition seem necessary. A perfect single-mode laser near threshold is theoretically described by the stochastic rotating wave Van der Pol oscillator (cf Risken 1970). This has been proved in recent years by very careful experiments (Arecchi and Degiorgio 1972, and the references therein, Gerhardt *et al* 1972). Neither experiment nor theory, supplemented by Agarwal's (1972) results on the susceptibility, show any indication of critical behaviour in the light field. From this one *must* conclude that the thermodynamic limit, with respect to the single laser mode, has not been reached. This evidently is related to the long coherence length of the emitted light corresponding to high spectral purity in only *one* mode, making these lasers effectively zero-dimensional systems (Grossmann and Richter 1971a).

The situation is different for the solid-state lasers cited by Dohm (1972b) which indeed may exhibit a sharp onset of lasing operation. Here the spatial mode structure is usually very complicated. In a first approximation one can simulate such a situation by our model of multi-mode operation. We therefore propose that the sharp onset of lasing operation, whenever it is observed, is related to the fact that *several* degrees of freedom of the light field *simultaneously* become unstable.

If a multi-mode operation exists with well separated individual thresholds then we do not expect a phase transition to occur. This seems to be confirmed by the experimental results of Lavine and Iannini (1965) on solid-state lasers. We further note that if there is a large number of modes with equal threshold values, then again one would expect a phase transition at each threshold.

Clearly, the investigation of these problems on a more fundamental basis requires one to consider the many-atom system coupled to a multi-wave light field, with a range of threshold values.

8. Discussion

We have presented exact analytical results for the statistics of a simple system of interacting waves in non-thermal equilibrium. The model is an extension of the stochastic rotating Van der Pol oscillator to a multi-wave situation in which N equivalent waves interact by neutral intensity coupling. The non-thermal equilibrium state is described by a single parameter a which plays the role of the reduced temperature. Analytic formulae have been given for the joint distribution functions $P(z(1), \ldots, z(N''))$ for the intensities $z(\mu)$ of a sub-group of N'' ($< N$) waves. The distribution function, the generating function, and the short-time counting distribution for the incoherent superimposition, I, of these waves have also been considered. The results have been discussed for a wide range of the parameters N, N'' and a.

One of the characteristic results is the thermalization of each individual wave if N is large. This thermalization is expected to be a general feature of coupled many-wave systems (it is one of the basic assumptions in Edwards' theory of turbulence (Edwards 1964, Edwards and McComb 1969)). It cannot be justified by involving the central limit theorem which refers to independent, that is physically non-interacting, stochastic variables. In fact, our model of interacting waves is an explicit counter example to the central limit theorem. The variance of I for example is smaller than that predicted by this theorem. Consequently, the interactions have reduced the fluctuations.

The exact results also allow one to discuss the thermodynamic limit $N \to \infty$. A mean-field phase transition is observed to occur in the order parameter $p \equiv \langle z \rangle^{1/2}$ at $a = 0$. The static susceptibility $\chi(a)$, associated with the thermodynamic limit of $\langle I \rangle$ (cf equation (20)), shows the Curie–Weiss behaviour of a ferromagnet in the paramagnetic phase.

It was further argued that the sharp onset of lasing operation at $a = 0$ (threshold) observed in certain lasers might be related to a spatially in-homogeneous multi-mode type light field structure in these lasers rather than due to the recently discovered phase transition of the atomic polarization which drives the light field (Dohm 1972a, b). However in Dohm's theory there is only one light mode and the idea of a phase transition in a system with only one degree of freedom seems self-contradictory.

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Appendix 1. Derivation of the model potential

One may arrive at the specific form of equation (1) for V by starting from a gain expansion around an unstable equilibrium.

The gain $\tilde{\alpha}(\nu)$ experienced by the wave with index ν can be written for sufficiently low intensities and assuming only intensity wave coupling in the form

$$\tilde{\alpha}(\nu) = b(\nu) - \sum_{\nu'=1}^{N} \tilde{c}(\nu, \nu')\tilde{z}(\nu'). \tag{A.1}$$

Here $b(\nu)$ is the linear gain in the time domain $\tilde{t}$, and $\tilde{c}$ is a positive symmetric coupling matrix which ensures that $\tilde{\alpha}(\nu)$ eventually goes to zero for increasing intensities $\tilde{z}(\nu')$. The condition $b(\nu) > 0$ means that the wave ν is unstable, at least for small intensity.

The diffusion coefficients $q(\nu)$ of the FP equation are taken to be independent of ν, that is $q(\nu) = q$. This condition is necessary so as to allow the integration of the stationary

FP equation (Graham 1973, chap 6.2). Under this condition one can introduce new scaled variables for the intensities $\tilde{z}(\nu)$ and time $\tilde{t}$ according to

$$z(\nu) = \left(\frac{\tilde{c}}{q}\right)^{1/2} \tilde{z}(\nu),$$

$$t = (\tilde{c}q)^{1/2}\tilde{t},$$

where $\tilde{c}$ is a typical value of a diagonal element of $\tilde{c}(\nu, \nu')$. The effective gain in the time domain t then becomes

$$\alpha(\nu) = \frac{b(\nu)}{d} - \sum_{\nu'} c(\nu, \nu')z(\nu'), \tag{A.2}$$

with $d = \sqrt{\tilde{c}q}$ and $c(\nu, \nu') = \tilde{c}(\nu, \nu')/\tilde{c}$. The potential V is then of the bilinear form

$$V = \tfrac{1}{4} \sum_{\nu,\nu'=1}^{N} (z(\nu) - a(\nu))c(\nu, \nu')(z(\nu') - a(\nu')), \tag{A.3}$$

where the quantities $a(\nu)$ satisfy

$$b(\nu) = \sum_{\nu'=1}^{N} c(\nu, \nu')a(\nu'). \tag{A.4}$$

They have the meaning of saturation intensities if they are greater than zero.

Equation (1) is finally obtained from (A.3) by requiring $c(\nu, \nu') \equiv 1$, that is neutral coupling between equivalent waves. From (A.4) it is seen that this implies $b(\nu) = b$, that is the thresholds of all waves coincide. Furthermore equation (A.3) implies that under this condition the pumping parameter a is given by $\Sigma\, a(\nu)/N$. Thus our model of a multi-mode system is such that there are many saturation values, $a(\nu)$, but only one threshold, namely $a = 0$.

In § 2, the viewpoint of fluctuations around an operating point has been emphasized and this leads in a much more general way to equation (1) for V (cf WL). It is now seen how both points of view may be related. If the matrix c is non-singular, equation (A.3) gives unique values for $a(\nu)$ for every given set $\{b(\nu')\}$. These $a(\nu)$ values then define the associated operating point. If c is singular the operating point is not uniquely defined.

Appendix 2. Lack of amplitude stabilization

We consider the case $N'' = 1$, that is the one-wave distribution $P_1(z)$, and prove that $P'_1(z_0) \neq 0$, for all z_0 in $[0; \infty)$, when $N \geqslant 2$. Consequently $P_1(z)$ must be a strictly decreasing function of z since $P_1(z)$ must approach zero as $z \to \infty$. There are two separate cases:

(i) $N = 2$. The differential equation (4) satisfied by $P_1(z)$ reduces to

$$2P''_1 = (2a - z)P'_1.$$

If at the point $z = z_0$, $P'_1(z_0) = 0$ then $P''_1(z_0)$ and all higher derivatives at this point are zero. Hence this would require $P_1(z)$ to be a constant, which is clearly impossible.

(ii) $N > 2$. In this case one has

$$P''_1(z_0) = (N - 2)P_1(z_0) \geqslant 0,$$

for all stationary points z_0. $P_1(z_0) = 0$ can be excluded since it implies $P_1(z) \equiv 0$.

Thus $P_1''(z_0) > 0$. It follows that $P_1(z)$ has at most minima. Topological reasons require that every maximum be followed by a minimum since $P_1(z \to \infty) \to 0$. Hence $P_1(z)$ can have neither minima nor maxima.

We next consider the joint distribution functions. The full distribution $P \propto \exp(-V)$, with V given by equation (1), is constant on the hyperplane $\Sigma\, z(v) = Na$ and decreases rapidly off this plane. After N' ($\geqslant 1$) contractions one obtains equation (4) for the distribution $P_c^{(N'')}$, which as a function of I admits the same discussion as that for $P_1(z)$. Thus one concludes that the multi-dimensional distribution functions $P(z(1), \ldots, z(N''))$ are also strictly decreasing functions of all its arguments. This means that the hyperplane $\Sigma\, z(v) = Na$, as the maximum surface for $N' = 0$, has moved into the unphysical region even for $N' = 1$.

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